low heatshield ablation rates in turbulent flow. Effects of Mach number on base pressure in the hypersonic, supersonic, and subsonic flight regimes were emphasized. Results of the present analysis revealed that the base pressure increases with increasing nose bluntness and cone angle, in agreement with previous correlations of nonablating data. The derived base drag coefficient increases with decreasing Mach number in supersonic flow and exhibits a maximum value near sonic conditions; a pronounced decrease in drag (increased base pressure) occurs as the Mach number is reduced slightly below unity.

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Displacement Field in the Nonlinear Theory of Shells

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THE present Note is concerned with a problem encountered in the definition of the displacement field for a cylindrical shell. Although the discussion is confined to cylindrical shells, it bears a somewhat fundamental character.

I. Using a convective coordinate system,[1] the Lagrangian strain tensor is defined as[1]

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{g}_{\alpha\beta} - g_{\alpha\beta}) \tag{1}$$

where α and β are indices of the shell surface and $\bar{g}$, g are the metric tensors of the deformed and the undeformed systems, respectively. Further, the relationship between the physical strain components and the strain tensor is given by

$$\varepsilon_{(\alpha)} = \left(1 + 2\frac{\gamma_{\alpha\alpha}}{g_{\alpha\alpha}} - 1\right)^{1/2} \tag{2}$$

In order to expand this expression in a power series, we have to assume that

$$\left|\frac{\gamma_{\alpha\alpha}}{g_{\alpha\alpha}}\right| \leq \frac{1}{2} \tag{3}$$

Thus,

$$\varepsilon_{(\alpha)} = \frac{\gamma_{\alpha\alpha}}{g_{\alpha\alpha}} - \frac{1}{2}\left(\frac{\gamma_{\alpha\alpha}}{g_{\alpha\alpha}}\right)^2 + \frac{1}{2}\left(\frac{\gamma_{\alpha\alpha}}{g_{\alpha\alpha}}\right)^3 \tag{4}$$

II. The Lagrangian strain tensor can be expressed in terms of the displacement vector using a body fixed coordinate system. Let this be transformed into a cylindrical coordinate system (r, φ, S). Assuming the validity of the Love-Kirchhoff assumptions we have

$$\gamma_{z\varphi} = \gamma_{xz} = 0 \tag{5}$$

$$\varepsilon_{zz} = 0 \tag{6}$$

and

$$\mathcal{W}(z) = \mathcal{W}(0) + z/r[d\mathcal{W}(0)/d(z/r)] \tag{7}$$

where $\mathcal{W}(z=0)$ is the displacement vector of a generic point on a distance $z=0$ of the middle surface. For the sake of simplicity and clarity, we will write the $\gamma_{\varphi\varphi}$ component only. However, the results are easily generalized. In Cartesian coordinates† we have

$$\gamma_{yy}\Big|_{u=0} = dv/dy + \frac{1}{2}[(dv/dy)^2 + (dw/dy)^2] \tag{8}$$

and transformed into cylindrical coordinates we obtain

$$\gamma_{\varphi\varphi}\Big|_{u=0} = (\dot{v} + w)\frac{1}{r} + \frac{1}{2r^2}[(w+\dot{v})^2 + (\dot{w} - v)^2] \tag{9}$$

where w and v are the displacement components of a generic point P of the middle surface of a cylindrical shell and $(\dot{\ }) = (\ )/d\varphi$.

It should be emphasized that in deriving this nonlinear expression, no assumption has been made on the magnitude of the displacement. The sole restriction on $\gamma_{\varphi\varphi}$ is that the strain should be very small.

III. We now look at the two well-known possible definitions of the displacement components (w,v) describing the deformation of the point P, see Fig. 1. One definition is made possible by postulating that the point first moves a radial distance w towards the center of the undeformed cylinder, then sideways another distance v which is perpendicular to w, and thus tangential to the undeformed reference surface at the point P. Thus, the exact geometrical expression connecting the displacement components with the physical strain defined by

$$\varepsilon_{\varphi} = (d\bar{S}_{\varphi} - dS_{\varphi})/dS_{\varphi} \tag{10}$$

is

$$\varepsilon_{\varphi} = \left[1 + \frac{1}{2}\left\{\frac{4}{r}(w+\dot{v}) + 2\left(\frac{\dot{v}+w}{r}\right)^2 + 2\left(\frac{\dot{w}-v}{r}\right)^2\right\}\right]^{1/2} - 1 \tag{11}$$

If the second term in brackets is defined as

$$\varepsilon_{\varphi} = [1 + \vartheta(w;v)]^{1/2} - 1 \tag{12}$$

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†See, for instance, V.V. Novozhilov, *Foundation of the Nonlinea Theory of Elasticity,* Graylock Press, pp. 4-15 and Eq. (1.22).

it is seen that ϑ is equal to $2\gamma_{\varphi\varphi}$. In order to expand $\mathcal{E}_\varphi$ in analogy to Eq. (4) in a power series, we assume that

$$|\vartheta(w;v)| \leq 1 \tag{13}$$

Thus

$$\mathcal{E}_\varphi = \frac{1}{2}\vartheta^2(w;v) + \ldots \tag{14}$$

Considering the first term only, we obtain

$$\mathcal{E}_\varphi = \tfrac{1}{2}\vartheta(w;v); \quad (\vartheta(w;v) << 1) \tag{15}$$

$$\mathcal{E}_\varphi = \gamma_{\varphi\varphi} \tag{16}$$

In other words, for a sufficiently small strain, the physical components are identical with the components of the strain tensor if we consider only the first term of the formal power expansion. However, a different situation arises if an alternative $w;v$ definition is considered, such as the definition frequently used by Flügge[2] and Pflüger.[3] In this case, it is postulated that the point P first moves sideways along the undeformed circular arc of the cylinder, a circumferential distance $\tilde{v}$, and then a radial distance $\tilde{w}$.[2,3] The exact strain displacement relationship is then given by (see Appendix II).

$$\tilde{\mathcal{E}}_\varphi = \left[\frac{(\dot{\tilde{w}})^2}{r} + \left\{\left(1 - \frac{\tilde{w}}{r}\right)\left(1 + \frac{\dot{\tilde{v}}}{r}\right)\right\}^2\right]^{1/2} - 1$$

$$= \left[1 + \tilde{\vartheta}(\tilde{w};\tilde{v})\right]^{1/2} - 1 \tag{17}$$

Proceeding as before, and retaining only the first term of the power expansion, we obtain

$$\tilde{\mathcal{E}}_\varphi = \tfrac{1}{2}\tilde{\vartheta}(\tilde{w};\tilde{v}); \quad (\tilde{\vartheta}(\tilde{w},\tilde{v}) << 1) \tag{18}$$

We note that $\tilde{\mathcal{E}}_\varphi$ is equal to $\gamma_{\varphi\varphi}$ if, and only if $\tilde{\mathcal{E}}_\varphi$ and $\gamma_{\varphi\varphi}$ are linearized. This is because, contrary to the case of a tangential v definition, $\tilde{\mathcal{E}}_\varphi$ includes third- and fourth-order terms in the displacement components and their derivatives.

IV. If we now neglect these third- and fourth-order terms, as in Refs. 2 and 3, we face the following problem. The assumption made in expanding $\tilde{\mathcal{E}}_\varphi$ in a power series, namely

$$|\tilde{\vartheta}(\tilde{w},\tilde{v})| < 1 \tag{19}$$

implies one of two possibilities: a) either every term $(\tilde{w},\tilde{v},\dot{\tilde{w}},\dot{\tilde{v}},c,\ldots)$ in $\tilde{\vartheta}(\tilde{w},\tilde{v})$ is sufficiently small so that the sum of all these terms is much smaller than unity, or b) the individual terms are not small, but only their sum is small compared with unity. It is obvious in case b) that we cannot simply ignore some of the terms without imposing a restriction on the magnitude of the displacement components and their derivatives.

The following expression, made up of second-order terms

$$\tilde{\mathcal{E}}_\varphi = \frac{1}{r}(\dot{\tilde{v}} - \tilde{w}) + \frac{1}{2r^2}(\dot{w}^2 - 2\dot{\tilde{v}}\tilde{w}) \tag{21}$$

is, therefore, more restrictive than $\mathcal{E}_\varphi = \gamma_{\varphi\varphi}$ since small displacement is assumed. The derivation of Eq. (21) is given in more detail in Appendix I.

V. The strain $\mathcal{E}_\varphi$, using a circumferential $\tilde{v}$ definition, does have several practical advantages. Consider the differential equations governing the quasicylindrical buckling‡ configuration of a circular cylindrical shell under hydrostatic pressure P. Using the circumferential $\tilde{v}$ definition, these are

$$p(\ddot{\tilde{w}} + \tilde{w}) + \frac{Et}{(1-\nu^2)r}(\tilde{w} - \dot{\tilde{v}})$$

$$+ \frac{Et^3}{12(1-\nu^2)r^3}(\ddot{\ddot{\tilde{w}}} + 2\ddot{\tilde{w}} + \tilde{w}) = 0 \tag{22}$$

$$\dot{\tilde{w}} - \ddot{\tilde{v}} = 0 \tag{23}$$

(see Refs. 2 or 3 for buckling that is independent of the x direction) where t is the shell thickness, E is the modulus of elasticity and ν is the Poisson ratio. We can now see the advantage of this formulation since a simple elimination by raising the order of the first differential equation results in the decoupling of the governing equation and we obtain

$$\ddot{\ddot{\tilde{\omega}}} + (2+\Lambda)\ddot{\tilde{\omega}} + (1+\Lambda)\tilde{\omega} = 0 \tag{24}$$

where

$$\Lambda = \frac{12p(1-\nu^2)r^3}{Et^3}; \quad \Lambda^c = 3 \tag{25}$$

We note that such a simple and elementary decoupling is not possible in the case of a tangential v definition. The equations corresponding to Eq. (25) in case of tangential v definitions are, for example

$$(\overset{IV}{\omega} + \overset{III}{\omega}) + \Lambda(\overset{II}{\omega} + \omega) - \alpha(\overset{I}{v} - \omega) = 0 \tag{26}$$

$$(\overset{III}{\omega} + \overset{II}{v}) + \alpha(\overset{II}{v} - \overset{I}{\omega}) = 0 \tag{27}$$

where $\alpha = (tr)^2/12$ and $(^I) = \mathrm{d}(\ )/\mathrm{d}\varphi$. The quasi-cylindrical buckling under a radially directed external pressure[4] ρ offers another example for the advantage of this formulation. If the loading energy of this type of follower is written down using the tangential v definition, a fairly complex expression would result. On the other hand, if we use the circumferential definition, the load energy would simply be

$$\pi_L = \int_0^{2\pi} p\,\tilde{\omega} r\,\mathrm{d}\varphi \tag{28}$$

In both cases, the branching load, of course, remains the same

$$\Lambda^c = 4.5 = p^c\,12(1-\nu^2)r^3/Et^3 \tag{29}$$

Using this argument, the conservativism of this type of loading was verified in Ref. 7.

Finally, let us consider an interesting connection between the measure for the changes in curvature and the two alternative displacement fields. The tensor for changes in curvature is

$$\mathcal{S}_{\alpha\beta} = \bar{b}_{\alpha\beta} - b_{\alpha\beta} \tag{30}$$

where $\bar{b}_{\alpha\beta}$, $b_{\alpha\beta}$ are the second fundamental tensor of the deformed and underformed middle surface of the shell respectively. In our particular case and retaining only linear terms, we obtain

$$\chi = (\ddot{\omega} + 3\dot{v} - 2\omega)/r^2 \tag{31}$$

This expression is inconsistent with the radial-tangential ($\omega - v$) definition and is a function of two unknowns (ω, v). In the case of a circumferential $\tilde{v}$ definition $[\tilde{w} = (\tilde{\omega},\tilde{v})]$ and using some results of the theory of curves, the exact expression is, however,

$$\tilde{\chi} = [(r+\tilde{\omega})(r+\tilde{\omega}+\ddot{\tilde{\omega}}) - 2\dot{\tilde{\omega}}^2][\dot{\tilde{\omega}}^2 + (r+\tilde{\omega})^2]^{-3/2} \tag{32}$$

‡This expression is frequently used in German literature[5] to describe the case when the wave length λ_x becomes very large so that buckling is rendered independent of the longitudinal direction ($\bar{\mu} = 0$) and becomes in the limiting case equivalent to a circular ring.[6]

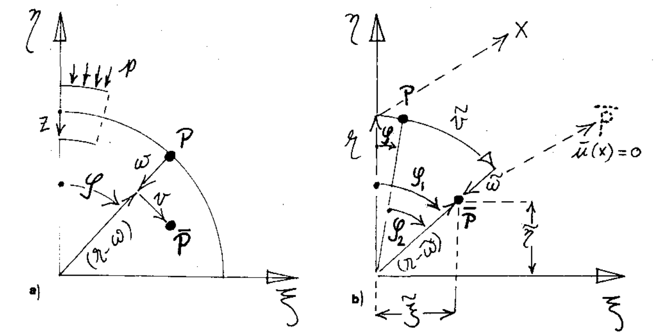

Fig. 1 Coordinate system ($r,\varphi,S=x/r$) and displacement field W of the circular cylindrical shell; a) in the case of tangential (vectorial) v definition $W(\omega,v,\bar{u})$, b) in the case of circumferential $\tilde{v}$ definition $\tilde{W}$ ($\tilde{\omega},\tilde{v},\tilde{u}$).

which is a function of only one unknown ($\tilde{\omega}$). On linearizing Eq. (32), the exact linear part is

$$\tilde{\chi}_L = (\ddot{\tilde{\omega}} + \tilde{\omega})/r^2 \tag{33}$$

which is obviously far simpler than the correspondant in the case of tangential v definition of Eq. (31). Similarly simple and accurate expressions, although approximate, can be obtained in the case of a tangential v definition only via Koiter-John's criteria of error estimation.[8-12,13]

Appendix 1 The Derivation of Eq. (21)

Expanding Eq. (17) we get

$$\bar{\varepsilon}_\varphi = 1 + \frac{1}{2}\Big[-\frac{2\tilde{\omega}}{r} + \Big(\frac{\tilde{\omega}}{r}\Big)^2 + \Big(\frac{\tilde{v}}{r}\Big)^2 - 4\frac{\tilde{v}^{\bullet}\tilde{\omega}}{r} + \Big(\frac{\dot{\tilde{\omega}}}{r}\Big)^2$$

+ third and fourth-order terms

$$-\frac{1}{8}\Big[-\frac{2\tilde{\omega}}{r} + \frac{2\tilde{v}}{r} + \text{second, third, and fourth-order terms}$$

$$+\text{terms of higher order}\Big]^2 + \ldots \tag{A1}$$

Since we have neglected third and fourth-order terms in the first bracket of Eq. (A1) and enforced a restriction on the magnitude of the displacement by assuming that every term must be sufficiently small so that the terms' sum remains smaller than unity (Eq. 20), we can consequently expand both brackets of Eq. (A1). Thus, we obtain

$$\bar{\varepsilon}_\varphi = \frac{1}{r}(\tilde{v} - \tilde{\omega}) + \frac{1}{2}\Big\{\frac{(\dot{\tilde{\omega}})^2}{r} - 2\frac{\tilde{\omega}\tilde{v}}{r^2}\Big\} \tag{A2}$$

which confirms Eq. (21).

Appendix 2 The Derivation of Eq. (17)

Following Fig. 1b we have

$$\varphi_1 = \varphi + \varphi_2 = \varphi + \tilde{v}/r$$

Thus

$$d\varphi_1 = d\varphi(1 + \frac{\dot{\tilde{v}}}{r});\ (\dot{\ }) = d(\ )/d\varphi \tag{A3}$$

We see further that

$$\tilde{\xi} = (r-\omega)\sin\varphi_1,\ \tilde{\eta} = (r-\omega)\cos\varphi_1$$

Thus

$$(\tilde{\xi}')^2 + (\tilde{\eta}')^2 = \tilde{\omega}'^2 + (r-\omega)^2;\ (\ ') = d(\ )/d\varphi_1 \tag{A4}$$

Since the length of an element $ds = r d\varphi$ after deformation is

$$\overline{dS}^2 = (d\tilde{\xi})^2 + (d\tilde{\eta})^2$$

we obtain

$$(\overline{dS}/d\varphi_1)^2 = (\tilde{\xi}')^2 + (\tilde{\eta}')^2 = \tilde{\omega}'^2 + (r-\tilde{\omega})^2 \tag{A5}$$

Considering Eqs. (A3) and (A4), the strain becomes

$$\tilde{\varepsilon}_{\varphi_1} = \frac{\overline{dS}}{dS} - 1 = \Big[\tilde{\omega}'^2 + (r-\tilde{\omega})^2\Big]^{1/2}\frac{d\varphi_1}{r d\varphi} - 1$$

$$= \frac{[\tilde{\omega}'^2 + (r-\tilde{\omega})^2]^{1/2}}{r - v'} - 1 \tag{A6}$$

With the aid of (A3) and the transformations

$$\tilde{\omega}' = d\tilde{\omega}/d\varphi_1 = \frac{d\tilde{\omega}}{d\varphi(1 + \tilde{v}^{\bullet}/r)} = \frac{\tilde{\omega}^{\bullet}}{(1 + \tilde{v}^{\bullet}/r)}$$

and $\tilde{v}' = \tilde{v}^{\bullet}/(1 + \tilde{v}^{\bullet}/r)$ the strain can be written in terms of the original co-ordinates as

$$\tilde{\varepsilon}_\varphi = \Big[\Big(\frac{\tilde{\omega}^{\bullet}}{r}\Big)^2 + \Big\{(1 - \frac{\tilde{\omega}}{r})(1 + \frac{\tilde{v}^{\bullet}}{r})\Big\}^2\Big]^{1/2} - 1$$

which confirms Eq. (17).

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Theoretical Analysis for the Burning of Solid Propellants

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Nomenclature

c = specific heat of solid

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